Tuesday, September 1, 2015

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Problem 41

Problem. Find the derivative of $f(x) = \ln 3x$.

Solution. Use the rule for ln and the Chain Rule for $3x$.

$$\begin{aligned} f'(x) &= \frac{1}{3x} \cdot 3 \\ &= \frac{3}{3x} \\ &= \frac{1}{x}. \end{aligned}$$

An alternative is to note that $f(x) = \ln 3 + \ln x$ and note that $\ln 3$ is a constant. Therefore,

$$f'(x) = \frac{1}{x}.$$

Problem 43

Problem. Find the derivative of $g(x) = \ln x^2$.

Solution. Use the rule for ln and the Chain Rule for x^2.

$$\begin{aligned} g'(x) &= \frac{1}{x^2} \cdot 2x \\ &= \frac{2x}{x^2} \\ &= \frac{2}{x}. \end{aligned}$$

An alternative is to note that $g(x) = 2 \ln x$. Therefore,

$$g'(x) = \frac{2}{x}.$$

Problem 45

Problem. Find the derivative of $y = (\ln x)^4$.

Solution. Use the Chain Rule.

$$\begin{aligned} y' &= 4(\ln x)^3 \cdot \frac{1}{x} \\ &= \frac{4(\ln x)^3}{x}. \end{aligned}$$

Problem 49

Problem. Find the derivative of $y = \ln\left(x\sqrt{x^2-1}\right)$.

Solution. Let's use the quicker method and rewrite this as $y = \ln x + \frac{1}{2}\ln(x^2-1)$. Then

$$\begin{aligned} y' &= \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2-1} \\ &= \frac{1}{x} + \frac{x}{x^2-1}. \end{aligned}$$

Problem 51

Problem. Find the derivative of $f(x) = \ln\left(\frac{x}{x^2+1}\right)$.

Solution. Rewrite this as $f(x) = \ln x - \ln(x^2+1)$. Then

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2+1}.$$

It is ok to leave the answer in that form. Or we could combine the two rational functions. Sometimes that pays off.

$$\begin{aligned} f'(x) &= \frac{1}{x} - \frac{2x}{x^2+1} \\ &= \frac{x^2+1}{x(x^2+1)} - \frac{2x^2}{x(x^2+1)} \\ &= \frac{x^2+1-2x^2}{x(x^2+1)} \\ &= \frac{-x^2+1}{x(x^2+1)}. \end{aligned}$$

Meh.

Problem 61

Problem. Find the derivative of $y = \ln|\sin x|$.

Solution. We know that the derivative of $\ln|x|$ is $\frac{1}{x}$ with domain $\{x \mid x \neq 0\}$. Use that rule and the Chain Rule.

$$\begin{aligned} y' &= \frac{\cos x}{\sin x} \\ &= \cot x. \end{aligned}$$

Problem 63

Problem. Find the derivative of $y = \ln \left| \dfrac{\cos x}{\cos x - 1} \right|$.

Solution. We can simplify this to $y = \ln|\cos x| - \ln|\cos x - 1|$. Then

$$\begin{aligned} y' &= \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} \\ &= -\tan x + \frac{\sin x}{\cos x - 1}. \end{aligned}$$

Problem 89

Problem. Use logarithmic differentiation to differentiate $y = x\sqrt{x^2+1}$.

Solution. Apply logarithms to both sides to get

$$\begin{aligned} \ln y &= \ln x\sqrt{x^2+1} \\ &= \ln x + \frac{1}{2}\ln(x^2+1). \end{aligned}$$

Then differentiate.

$$\begin{aligned} \frac{y'}{y} &= \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} \\ &= \frac{1}{x} + \frac{x}{x^2+1}. \end{aligned}$$

Then multiply by y to get y'.

$$\begin{aligned} y' &= y\left(\frac{1}{x} + \frac{x}{x^2+1}\right) \\ &= x\sqrt{x^2+1}\left(\frac{1}{x} + \frac{x}{x^2+1}\right) \\ &= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}. \end{aligned}$$

Problem 91

Problem. Use logarithmic differentiation to differentiate $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}$.

Solution. Apply logarithms to both sides to get

$$\begin{aligned}\ln y &= \ln\left(\frac{x^2\sqrt{3x-2}}{(x+1)^2}\right)\\ &= 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1).\end{aligned}$$

Then differentiate.

$$\begin{aligned}\frac{y'}{y} &= \frac{2}{x} + \frac{1}{2}\cdot\frac{3}{3x-2} - \frac{2}{x+1}\\ &= \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}.\end{aligned}$$

Then multiply by y to get y'.

$$\begin{aligned}y' &= y\left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right)\\ &= \frac{x^2\sqrt{3x-2}}{(x+1)^2}\cdot\left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right).\end{aligned}$$

That can be simplified, but it is not necessary and probably not worth it.

Problem 93

Problem. Use logarithmic differentiation to differentiate $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}$.

Solution. Apply logarithms to both sides to get

$$\begin{aligned}\ln y &= \ln\left(\frac{x(x-1)^{3/2}}{\sqrt{x+1}}\right)\\ &= \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1).\end{aligned}$$

Then differentiate.

$$\frac{y'}{y} = \frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}.$$

Then multiply by y to get y'.

$$\begin{aligned}y' &= y\left(\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right)\\ &= \frac{x(x-1)^{3/2}}{\sqrt{x+1}}\cdot\left(\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right).\end{aligned}$$

Problem 103

Problem. The term t of a \$200,000 home mortgage as 7.5% interest can be approximated by

$$t = 13.375 \ln\left(\frac{x}{x-1250}\right),$$

where x is the monthly payment. [I will do part (d).] Find the instantaneous rates of change of t with respect to x when $x = 1398.43$ and $x = 1611.19$.

Solution. First, rewrite t as

$$t = 13.375(\ln x - \ln(x - 1250))$$

and then find the derivative.

$$\frac{dt}{dx} = 13.375\left(\frac{1}{x} - \frac{1}{x-1250}\right).$$

Evaluate at $x = 1398.43$ and at $x = 1611.19$.

$$\begin{aligned}\frac{dt}{dx}(1398.43) &= 13.375\left(\frac{1}{1398.43} - \frac{1}{148.43}\right)\\ &= -0.0805.\\ \frac{dt}{dx}(1611.19) &= 13.375\left(\frac{1}{1611.19} - \frac{1}{361.19}\right)\\ &= -0.0287.\end{aligned}$$

Problem 107

Problem. A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix*. The equation of this path is

$$y = 10\ln\left(\frac{10+\sqrt{100-x^2}}{x}\right) - \sqrt{100-x^2}.$$

[I will do parts (b) and (c).] Find the slopes of this path when $x = 5$ and $x = 9$. What does the slope of the path approach as $x \to 10$?

Solution. Rewrite the function as

$$y = 10\left(\ln(10+\sqrt{100-x^2}) - \ln x\right) - \sqrt{100-x^2}$$

and differentiate.

$$\begin{aligned}
y' &= 10\left(\frac{\frac{d}{dx}(10+\sqrt{100-x^2})}{10+\sqrt{100-x^2}} - \frac{1}{x}\right) + \frac{x}{\sqrt{100-x^2}} \\
&= 10\left(\frac{-\frac{x}{\sqrt{100-x^2}}}{10+\sqrt{100-x^2}} - \frac{1}{x}\right) + \frac{x}{\sqrt{100-x^2}} \\
&= 10\left(-\frac{x}{\sqrt{100-x^2}(10+\sqrt{100-x^2})} - \frac{1}{x}\right) + \frac{x}{\sqrt{100-x^2}}.
\end{aligned}$$

It will pay to simplify this. Get a common denominator of

$$x\sqrt{100-x^2}(10+\sqrt{100-x^2}).$$

Gather the terms in the numerator and discover that the fraction reduces to

$$y' = -\frac{\sqrt{100-x^2}}{x}.$$

Evaluate at $x = 5$ and $x = 9$.

$$\begin{aligned}
y'(5) &= -\frac{\sqrt{75}}{5} = -\sqrt{3}. \\
y'(9) &= -\frac{\sqrt{19}}{9}.
\end{aligned}$$

Now it is easy to take the limit as $x \to 10$.

$$\lim_{x\to 10^+}\left(-\frac{\sqrt{100-x^2}}{x}\right) = 0.$$